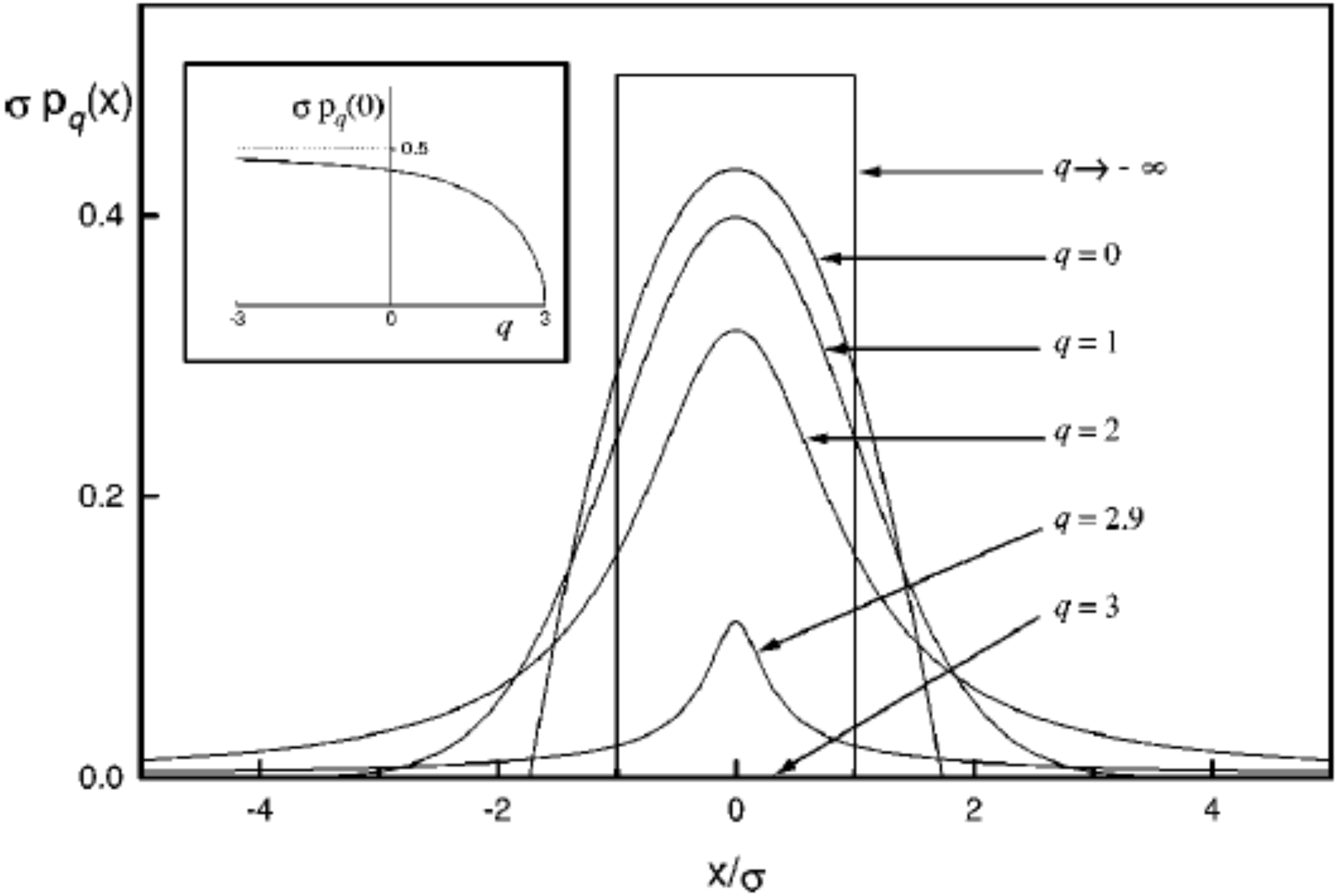
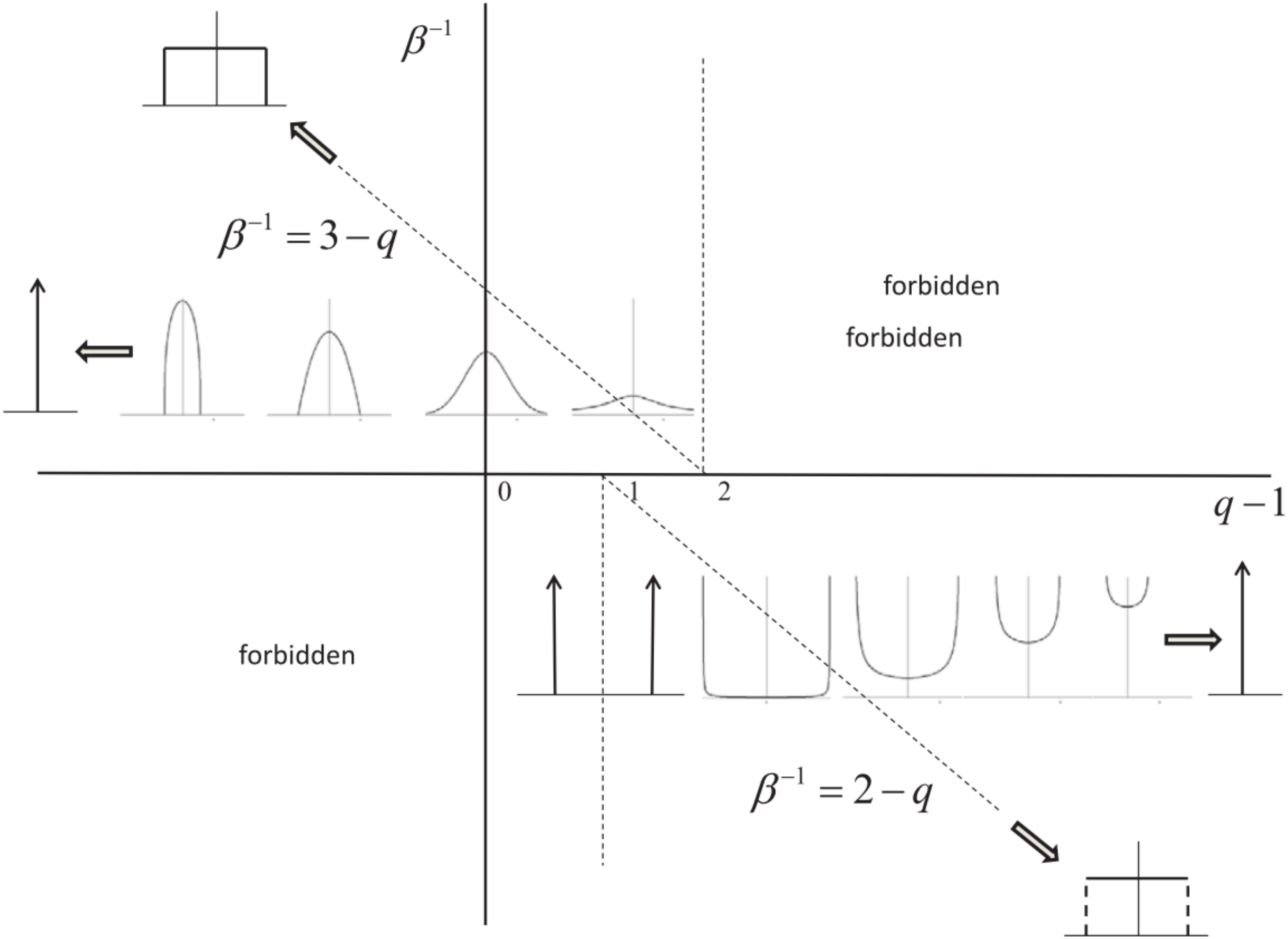
q-GAUSSIANS: $p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \frac{1}{\left[1+(q-1)\,(x/\sigma)^2\right]^{\frac{1}{q-1}}}$ $(q<3)$



D. Prato and C. T., Phys Rev E **60**, 2398 (1999)



A. Rodriguez and C. T. , J Stat Mech P12027 (2014)

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{ix\xi[f(x)]^{q-1}} f(x)\, dx$$

$(q \geq 1)$

(nonlinear!)

For q<1 see K.P. Nelson and S. Umarov, Physica A **389**, 2157 (2010)

$$q-FourierTransform\left[\frac{\sqrt{\beta}}{C_q}e_q^{-\beta\, t^2}\right]=e_{q_1}^{-\beta_1\,\omega^2}$$

where $$q_1=\frac{1+q}{3-q}$$ $$(q_1\geq q\geq 1)$$

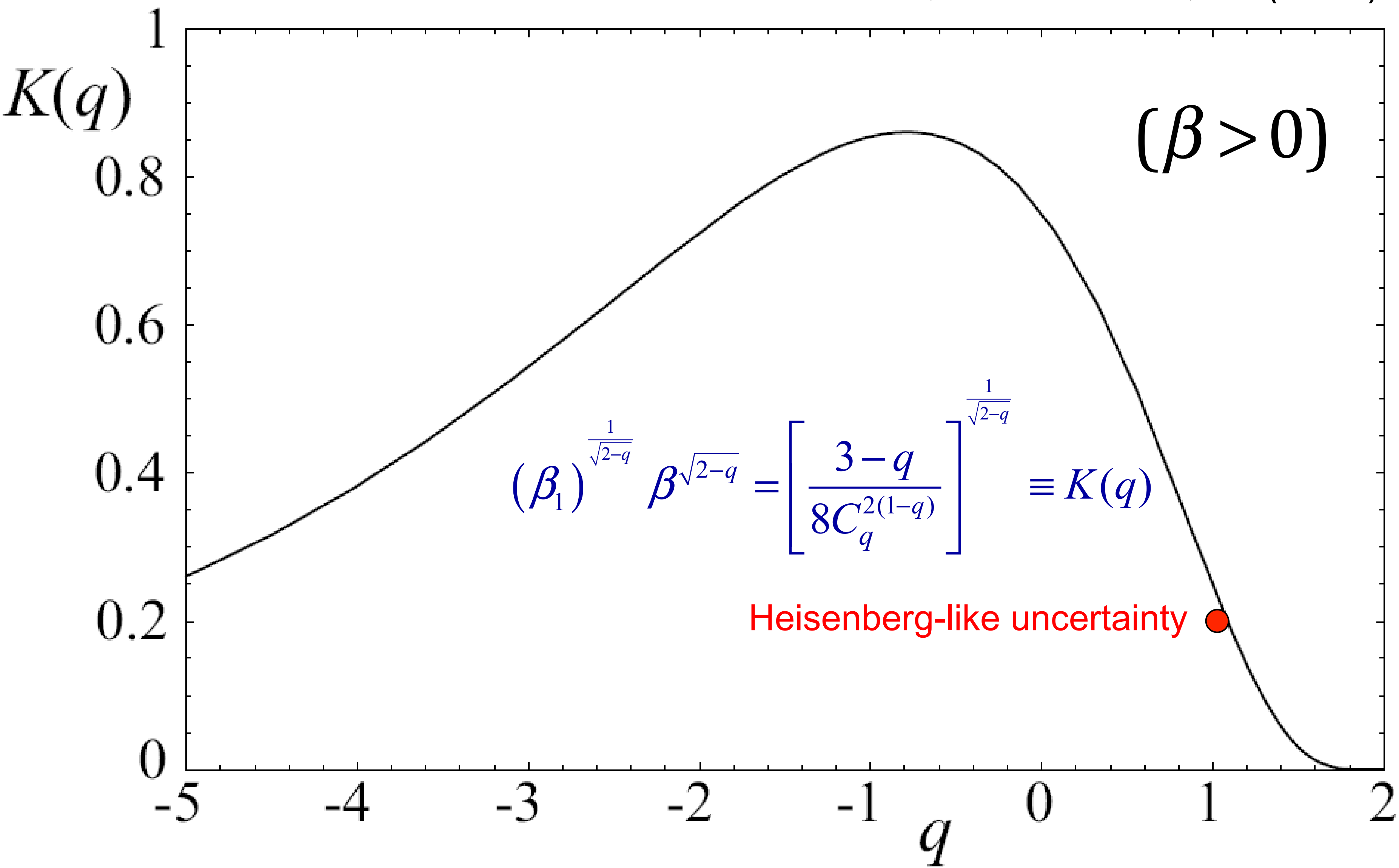
invertible

and $$\beta_1=\frac{3-q}{8\beta^{2-q}C_q^{2(1-q)}} \Leftrightarrow (\beta_1)^{\frac{1}{\sqrt{2-q}}}\beta^{\sqrt{2-q}}=\left[\frac{3-q}{8C_q^{2(1-q)}}\right]^{\frac{1}{\sqrt{2-q}}}\equiv K(q)$$

with

$$C_q=\begin{cases}\dfrac{2\sqrt{\pi}\,\Gamma\left(\dfrac{1}{q-1}\right)}{(3-q)\sqrt{(1-q)}\,\Gamma\left(\dfrac{3-q}{2(1-q)}\right)} & if\quad q<1\\[2ex] \sqrt{\pi} & if\quad q=1\\[2ex] \dfrac{\sqrt{\pi}\,\Gamma\left(\dfrac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\dfrac{1}{q-1}\right)} & if\quad 1<q<3\end{cases}$$

S.M.D. Queiros and C. T., AIP Proc **978**, 21 (2007)



Consequently, for fixed $q < 2$,

when $\beta > 0$ increases (decreases), $\beta_1 > 0$ decreases (increases)

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-independence:

Two random variables X [*with density* $f_X(x)$] *and* Y [*with density* $f_Y(y)$] *having zero* $q-$*mean values are said* q*-independent if*

$$F_q[\mathrm{X+Y}](\xi) = F_q[\mathrm{X}](\xi) \otimes_{\frac{1+q}{3-q}} F_q[\mathrm{Y}](\xi) \quad ,$$

i.e., if

$$\int_{-\infty}^{\infty} dz\ e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\ e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_{(1+q)/(3-q)} \left[\int_{-\infty}^{\infty} dy\ e_q^{iy\xi} \otimes_q f_Y(y)\right],$$

with

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ h(x,y)\ \delta(x+y-z) = \int_{-\infty}^{\infty} dx\ h(x,z-x) = \int_{-\infty}^{\infty} dy\ h(z-y,y)$$

where $h(x,y)$ *is the joint density.*

$$q\text{-}independence\ means \begin{cases} independence & if\ \ q=1\ ,\ i.e.,\ h(x,y)=f_X(x)\,f_Y(y) \\ global\ correlation & if\ \ q\neq 1\ ,\ i.e.,\ h(x,y)\neq f_X(x)\,f_Y(y) \end{cases}$$

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033502 (2010)

Generalization of symmetric α-stable Lévy distributions for $q>1$

Sabir Umarov,[1,a)] Constantino Tsallis,[2,3,b)] Murray Gell-Mann,[3,c)] and Stanly Steinberg[4,d)]
[1]*Department of Mathematics, Tufts University, Medford, Massachusetts 02155, USA*
[2]*Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
[3]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*
[4]*Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, USA*

(Received 10 November 2009; accepted 4 January 2010; published online 3 March 2010)

See also: H.J. Hilhorst, JSTAT P10023 (2010)

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A. Plastino and M.C. Rocca, Physica A and Milan J Math (2012)

A. Plastino and M.C. Rocca, Physica A **392**, 3952 (2013)

S. Umarov and C. T., J Phys A **49**, 415204 (2016)

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ *-scaled attractor* $\mathrm{F}(x)$ *when summing* $N \to \infty$ q*-independent identical random variables with symmetric distribution* $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \frac{1+q}{3-q} \right)$

	$q=1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathrm{F}(x) =$ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathrm{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \lvert x \rvert << x_c(q,2) \\ f(x) \sim C_q / \lvert x \rvert^{2/(q-1)} & \text{if } \lvert x \rvert >> x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathrm{F}(x) =$ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *behavior* $L_\alpha(x) \sim \begin{cases} G(x) \quad \text{if } \lvert x \rvert << x_c(1,\alpha) \\ f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \quad \text{if } \lvert x \rvert >> x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathrm{F}(x) = L_{q,\alpha}$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\,\alpha}(x) \sim C^*_{q,\alpha} / \lvert x \rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \textit{(intermediate regime)} \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\,2}(x) \sim C^L_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \textit{(distant regime)} \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg J Math Phys 51, 033502 (2010)

Physics Reports 478 (2009) 1–69

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Physics Reports

journal homepage: www.elsevier.com/locate/physrep

The large deviation approach to statistical mechanics

Hugo Touchette *

School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK

ARTICLE INFO

Article history:
Accepted 29 May 2009
Available online 6 June 2009
editor: I. Procaccia

PACS:
05.20.-y
65.40.Gr
02.50.-r
05.40.-a

ABSTRACT

The theory of large deviations is concerned with the exponential decay of probabilities of large fluctuations in random systems. These probabilities are important in many fields of study, including statistics, finance, and engineering, as they often yield valuable information about the large fluctuations of a random system around its most probable state or trajectory. In the context of equilibrium statistical mechanics, the theory of large deviations provides exponential-order estimates of probabilities that refine and generalize Einstein's theory of fluctuations. This review explores this and other connections between large deviation theory and statistical mechanics, in an effort to show that the mathematical language of statistical mechanics is the language of large deviation theory. The first part of the review presents the basics of large deviation theory, and works out many of its classical applications related to sums of random variables and Markov processes. The second part goes through many problems and results of statistical mechanics, and shows how these can be formulated and derived within the context of large deviation theory. The problems and results treated cover a wide range of physical systems, including equilibrium many-particle systems, noise-perturbed dynamics, nonequilibrium systems, as well as multifractals, disordered systems, and chaotic systems. This review also covers

	PHYSICS (Statistical mechanics)	**MATHEMATICS** (Large deviation theory)
$q=1$ (quasi-independent)	$p_N \propto e^{-\beta H_N} = e^{-\left[\beta \frac{H_N}{N}\right] N}$	$P_N(x) \sim e^{-N\, r(x)}$ $(N \to \infty)$
$q>1$ (strongly correlated)	$p_N \propto e_q^{-\beta_q H_N} = e_q^{-\left[(\beta_q N^*) \frac{H_N}{N N^*}\right] N}$	$P_N(x) \sim e_q^{-N\, r_q(x)}$ $(N \to \infty)$

G. Ruiz and C. T., Phys Lett A **376**, 2451 (2012)

Physics Letters A 376 (2012) 2451–2454

Contents lists available at SciVerse ScienceDirect

Physics Letters A

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Towards a large deviation theory for strongly correlated systems

Guiomar Ruiz [a,b,*], Constantino Tsallis [a,c]

[a] *Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil*
[b] *Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Madrid, Pza. Cardenal Cisneros s.n., 28040 Madrid, Spain*
[c] *Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA*

ARTICLE INFO

Article history:
Received 12 March 2012
Received in revised form 20 June 2012
Accepted 22 June 2012
Available online 28 June 2012
Communicated by C.R. Doering

ABSTRACT

A large-deviation connection of statistical mechanics is provided by N independent binary variables, the $(N \to \infty)$ limit yielding Gaussian distributions. The probability of $n \neq N/2$ out of N throws is governed by e^{-Nr}, r related to the entropy. Large deviations for a strong correlated model characterized by indices (Q, γ) are studied, the $(N \to \infty)$ limit yielding Q-Gaussians ($Q \to 1$ recovers a Gaussian). Its large deviations are governed by $e_q^{-Nr_q}$ ($\propto 1/N^{1/(q-1)}$, $q > 1$), $q = (Q-1)/(\gamma[3-Q]) + 1$. This illustration opens the door towards a large-deviation foundation of nonextensive statistical mechanics.

Physics Letters A 377 (2013) 491–495

Contents lists available at SciVerse ScienceDirect

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Comment

Reply to Comment on "Towards a large deviation theory for strongly correlated systems"

Guiomar Ruiz [a,b,*], Constantino Tsallis [a,c]

[a] Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brazil
[b] Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Madrid, Pza. Cardenal Cisneros s.n., 28040 Madrid, Spain
[c] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

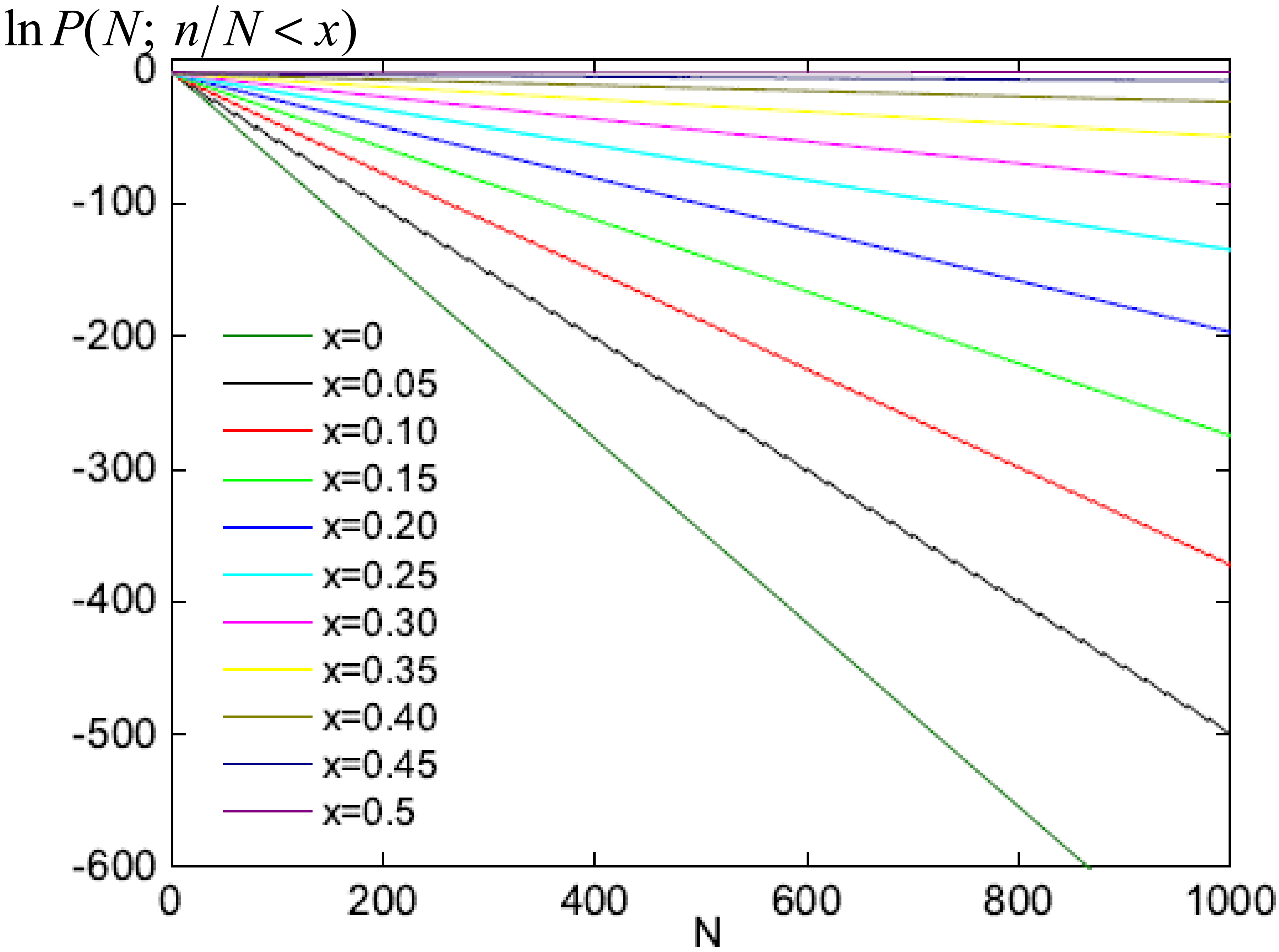
ARTICLE INFO

Article history:
Received 22 November 2012
Accepted 2 December 2012
Available online 14 December 2012
Communicated by C.R. Doering

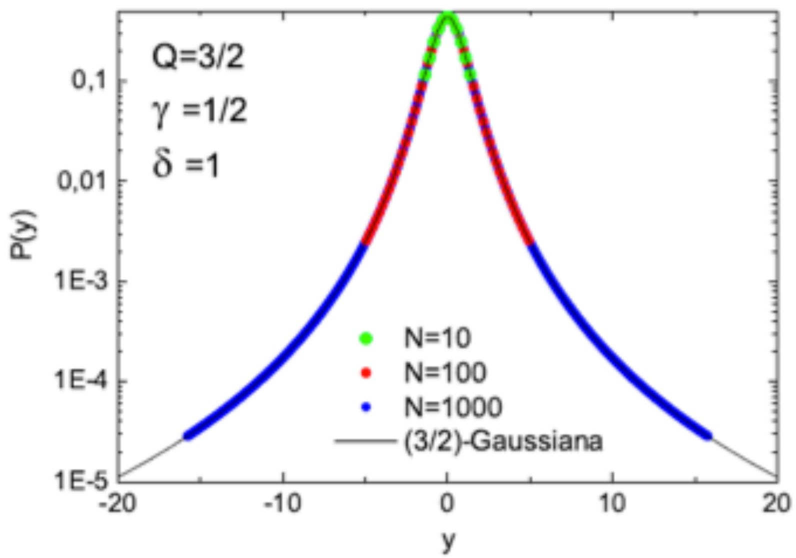
Keywords:
Probability theory
Statistical mechanics
Entropy

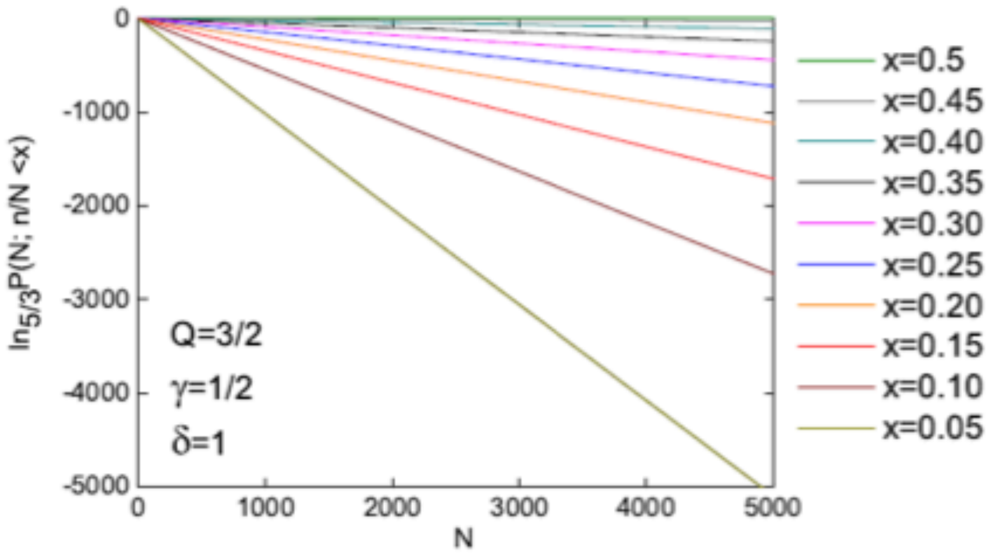
ABSTRACT

The computational study commented by Touchette opens the door to a desirable generalization of standard large deviation theory for special, though ubiquitous, correlations. We focus on three inter-related aspects: (i) numerical results strongly suggest that the standard exponential probability law is asymptotically replaced by a power-law dominant term; (ii) a subdominant term appears to reinforce the thermodynamically extensive entropic nature of q-generalized rate function; (iii) the correlations we discussed, correspond to Q-Gaussian distributions, differing from Lévy's, except in the case of Cauchy–Lorentz distributions. Touchette has agreeably discussed point (i), but, unfortunately, points (ii) and (iii) escaped to his analysis. Claiming the absence of connection with q-exponentials is unjustified.

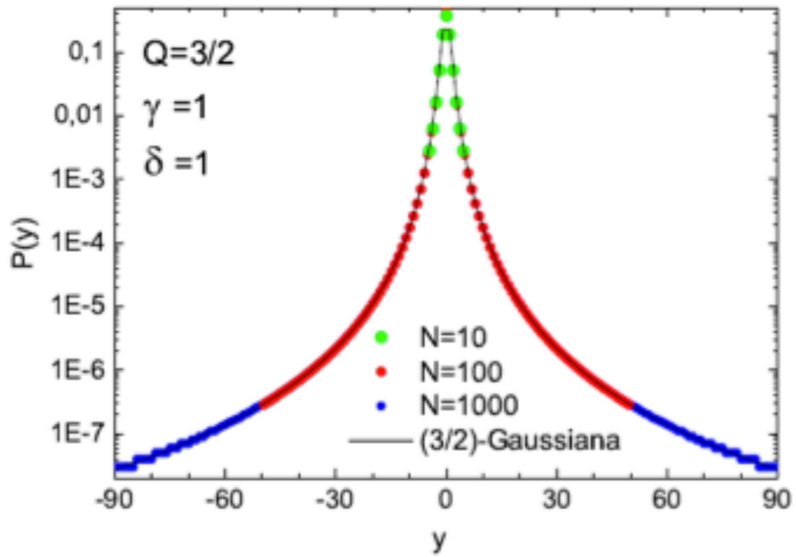


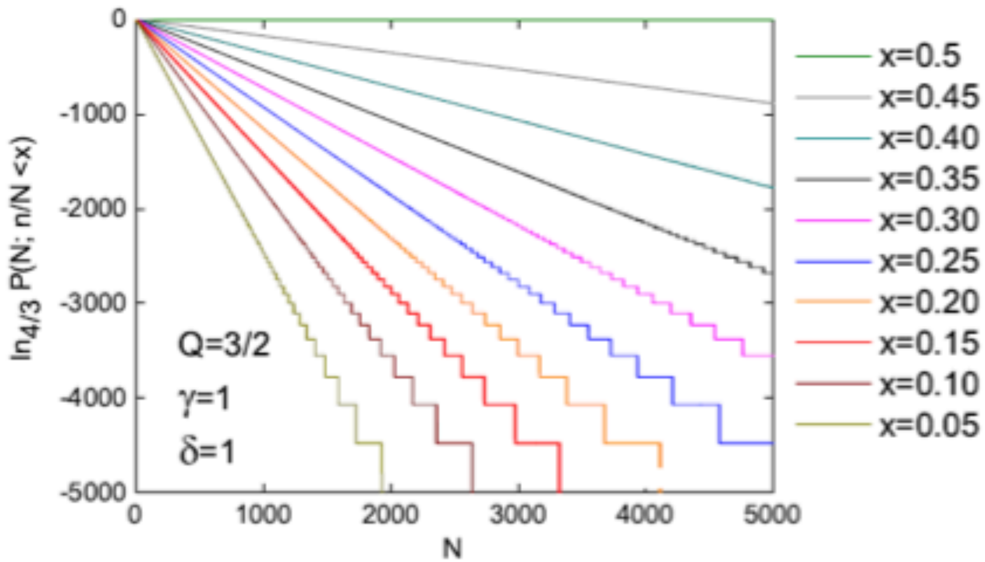
G. Ruiz and C. T., Phys Lett A **376**, 2451 (2012)



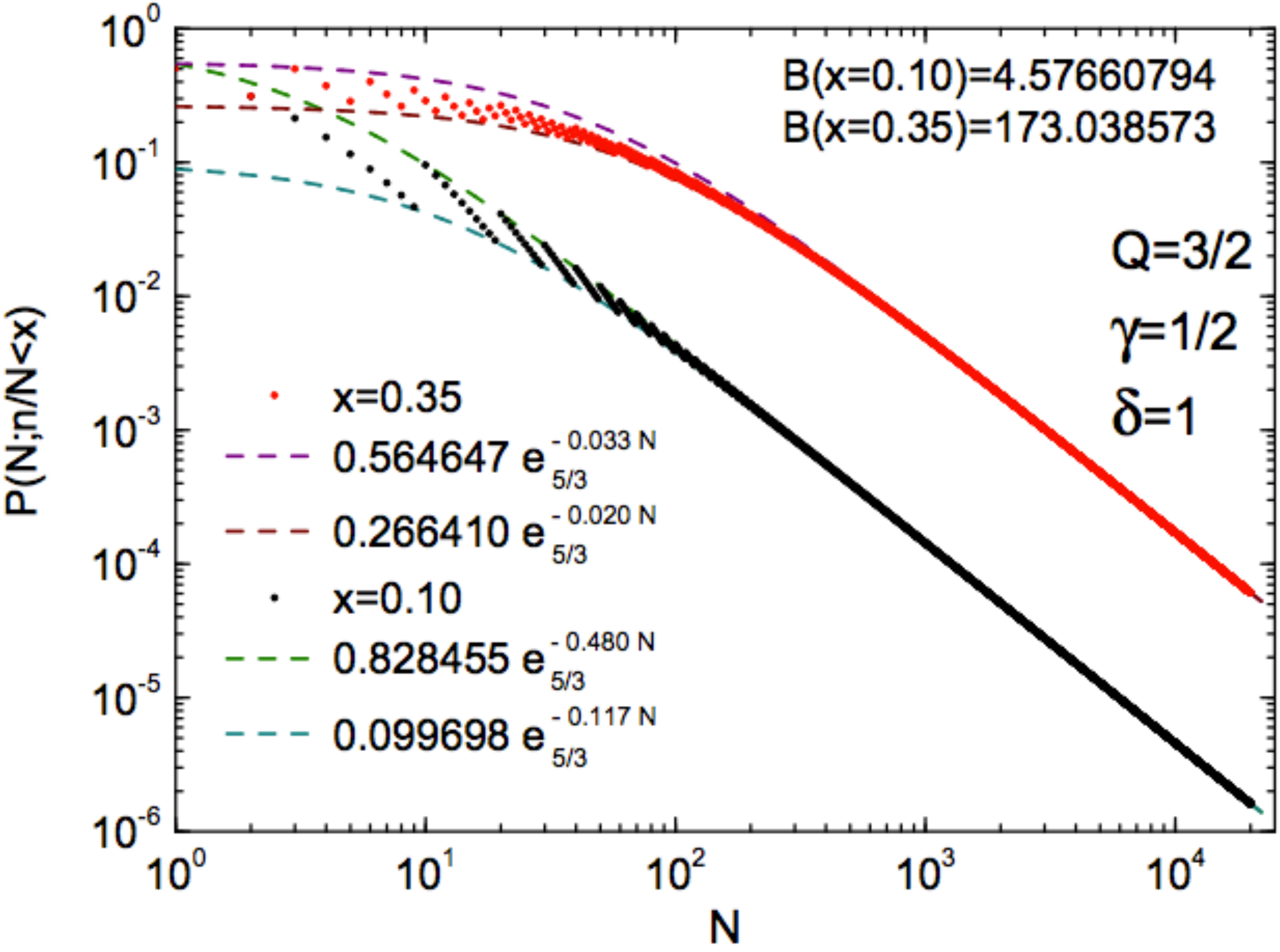


$$q = 1 + \frac{Q-1}{\gamma(3-Q)}$$

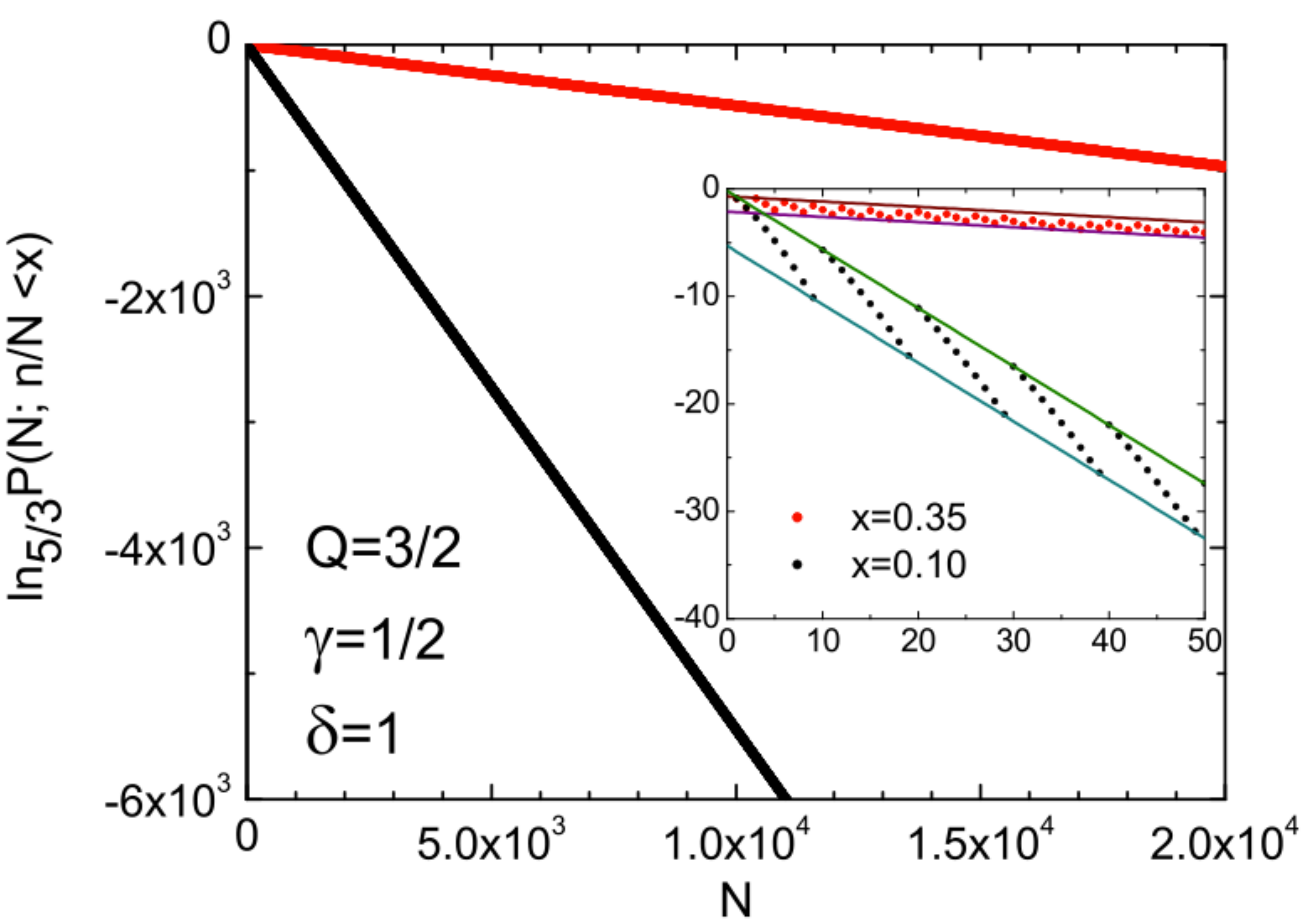




G. Ruiz and C. T., Phys Lett A **376**, 2451 (2012)



G. Ruiz and C. T., Phys Lett A **377**, 491 (2013)



G. Ruiz and C. T., Phys Lett A **377**, 491 (2013)

REVIEWS OF MODERN PHYSICS VOLUME 27, NUMBER 3 JULY, 1955

Foundations of Statistical Mechanics*

D. TER HAAR

Department of Natural Philosophy, St. Salvator's College, St. Andrews, Scotland

The first man to use a truly statistical approach was Boltzmann [1877] and at that point kinetic theory changed into statistical mechanics even though it was another twenty odd years before Gibbs coined the expression.

One might argue that the proof of the pudding is in the eating, and that the fact that statistical mechanics has been able to predict accurately and successfully the behavior of physical systems under equilibrium conditions—and even under certain circumstances in non-equilibrium conditions[4]—should be a sufficient justification for the methods used.

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SCIENTIFIC REPORTS

OPEN

The standard map: From Boltzmann-Gibbs statistics to Tsallis statistics

Received: 10 December 2015
Accepted: 09 March 2016
Published: 23 March 2016

Ugur Tirnakli[1,*] & Ernesto P. Borges[2,3,*]

As well known, Boltzmann-Gibbs statistics is the correct way of thermostatistically approaching ergodic systems. On the other hand, nontrivial ergodicity breakdown and strong correlations typically drag the system into out-of-equilibrium states where Boltzmann-Gibbs statistics fails. For a wide class of such systems, it has been shown in recent years that the correct approach is to use Tsallis statistics instead. Here we show how the dynamics of the paradigmatic conservative (area-preserving) standard map exhibits, in an exceptionally clear manner, the crossing from one statistics to the other. Our results unambiguously illustrate the domains of validity of both Boltzmann-Gibbs and Tsallis statistical distributions. Since various important physical systems from particle confinement in magnetic traps to autoionization of molecular Rydberg states, through particle dynamics in accelerators and comet dynamics, can be reduced to the standard map, our results are expected to enlighten and enable an improved interpretation of diverse experimental and observational results.

STANDARD MAP (Chirikov 1969)

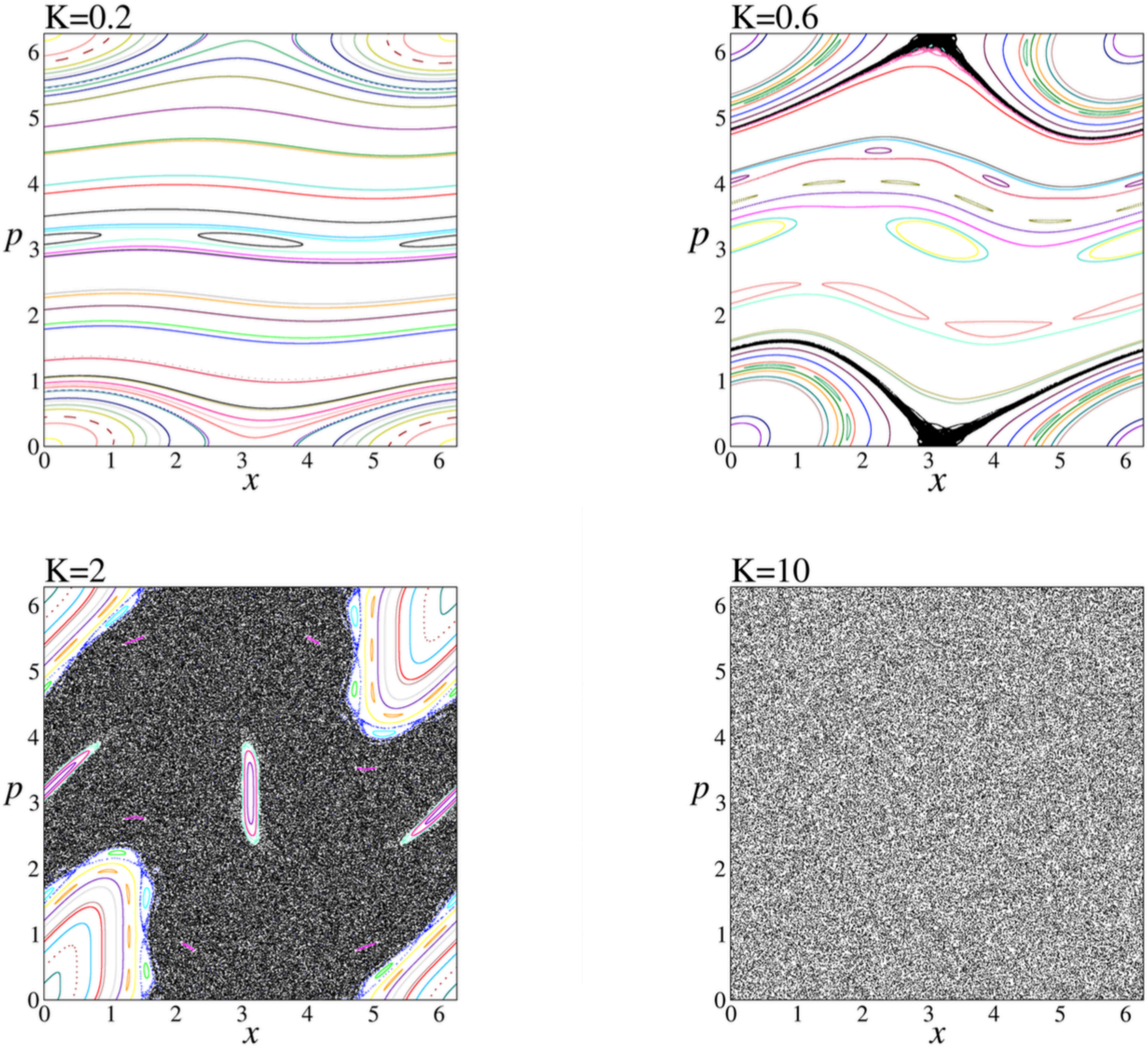
$$p_{i+1} = p_i - K \sin x_i \quad (\mathrm{mod}\ 2\pi)$$

$$x_{i+1} = x_i + p_{i+1} \quad (\mathrm{mod}\ 2\pi)$$

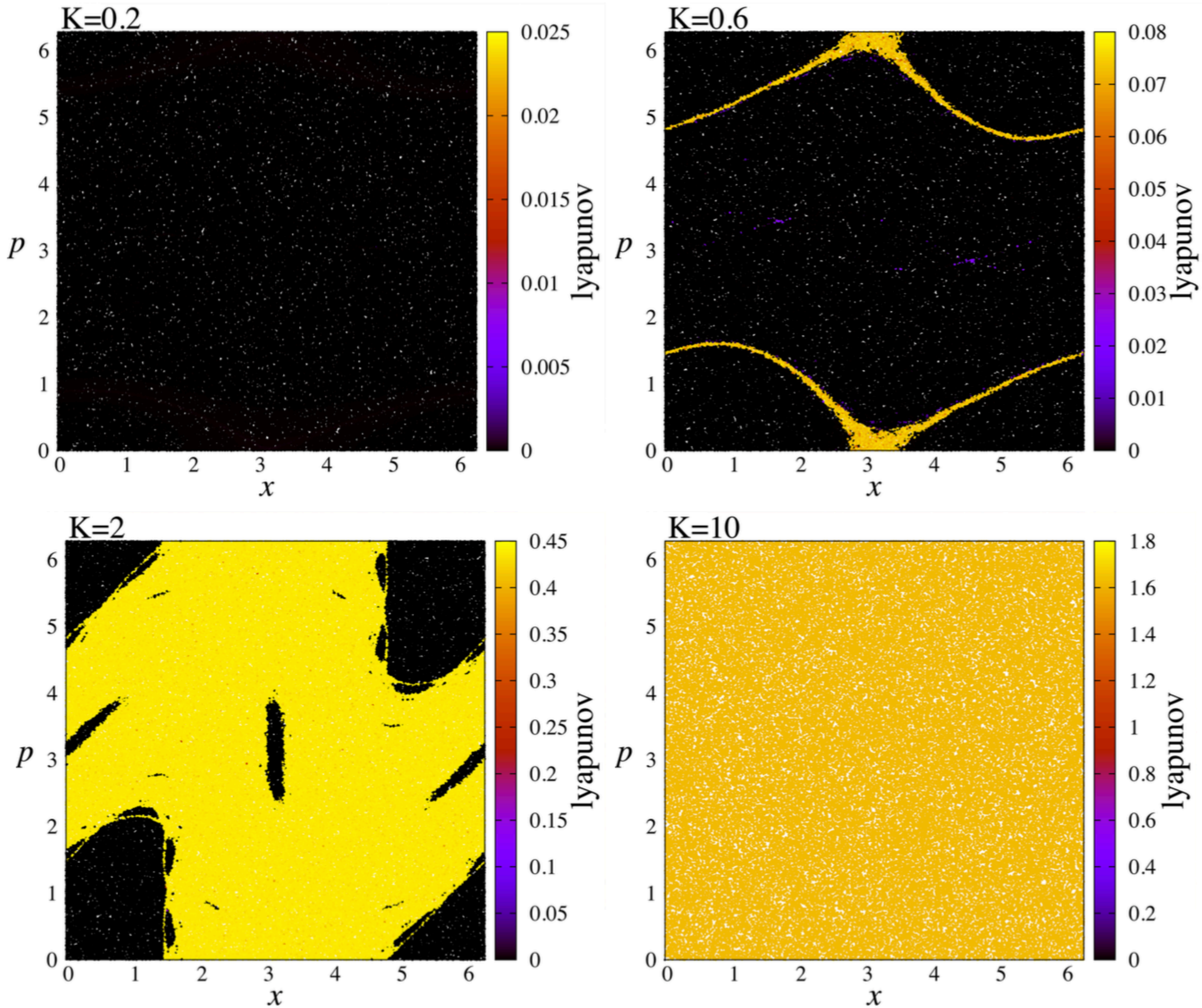
$$(i = 0,1,2,\ldots)$$

(area-preserving)

Particle confinement in magnetic traps,
particle dynamics in accelerators,
comet dynamics,
ionization of Rydberg atoms,
electron magneto-transport



Tirnakli and Borges
Nature / Scientific Reports **6**, 23644 (2016)



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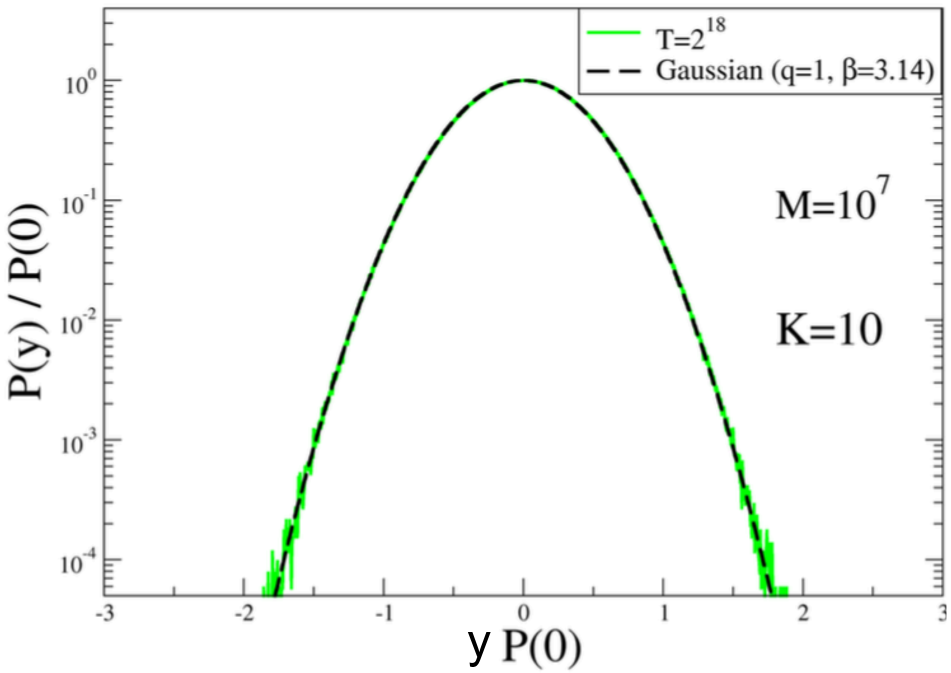
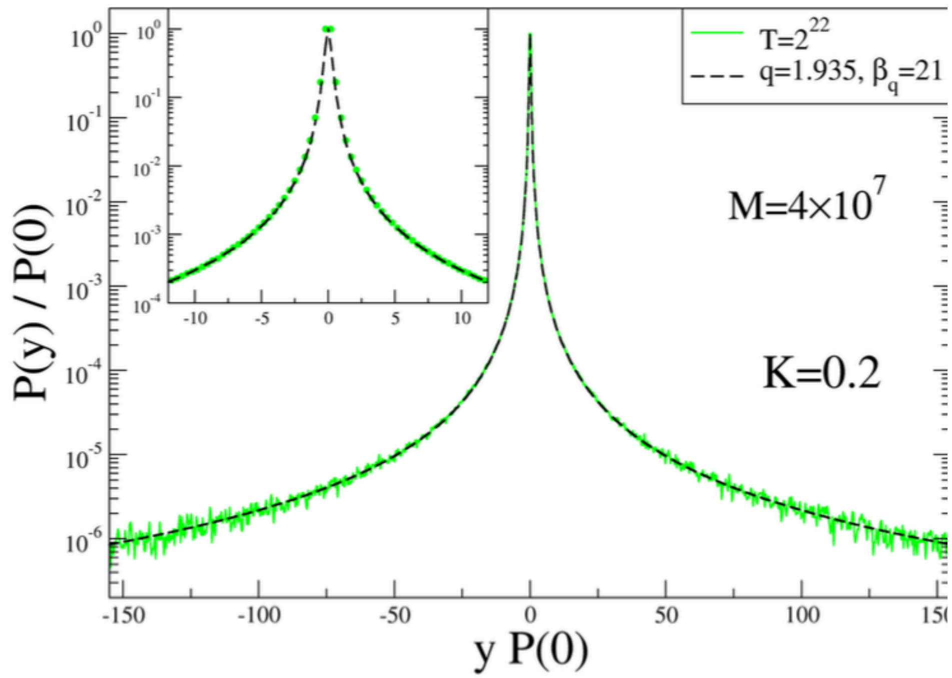
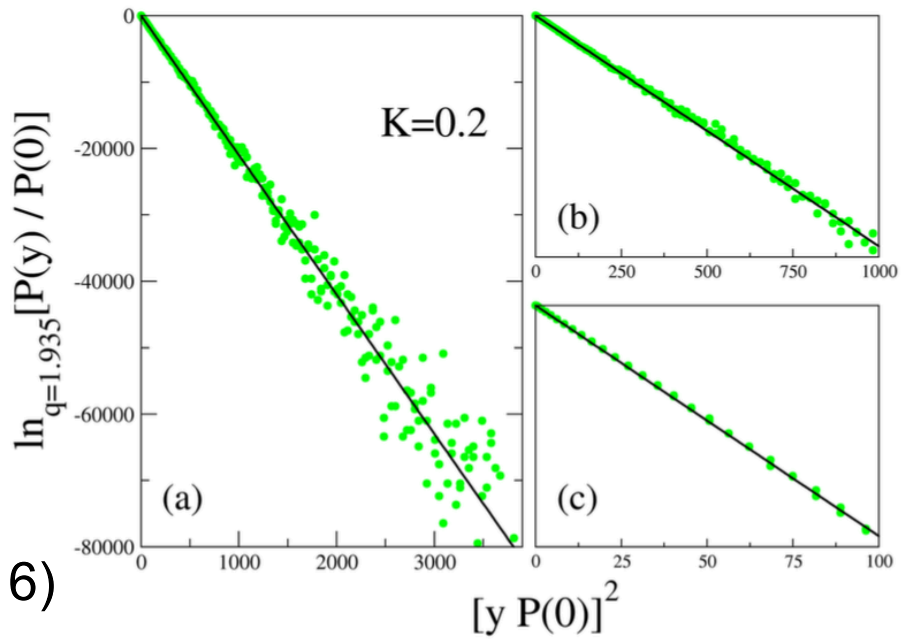
Nature / Scientific Reports **6**, 23644 (2016)

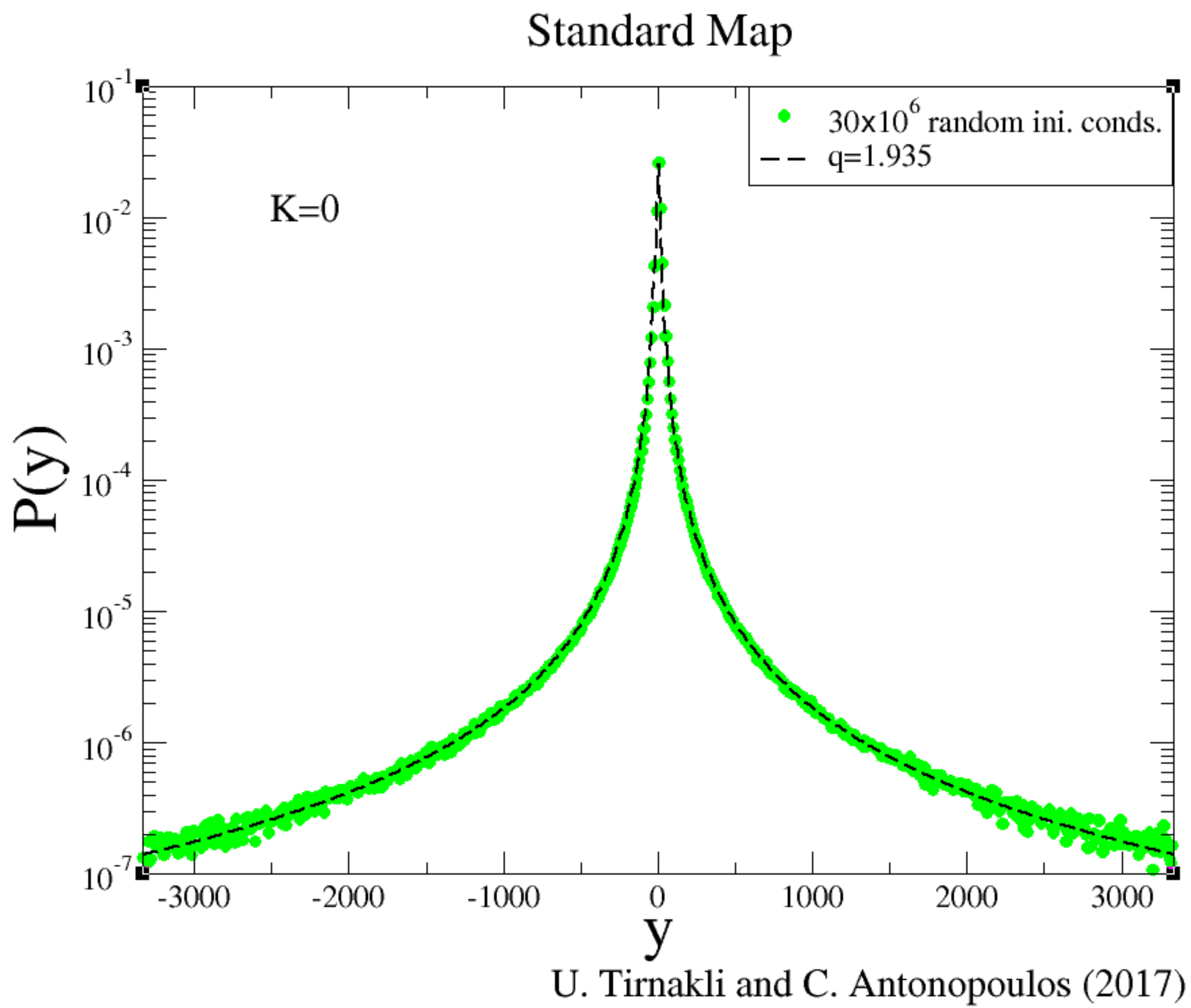




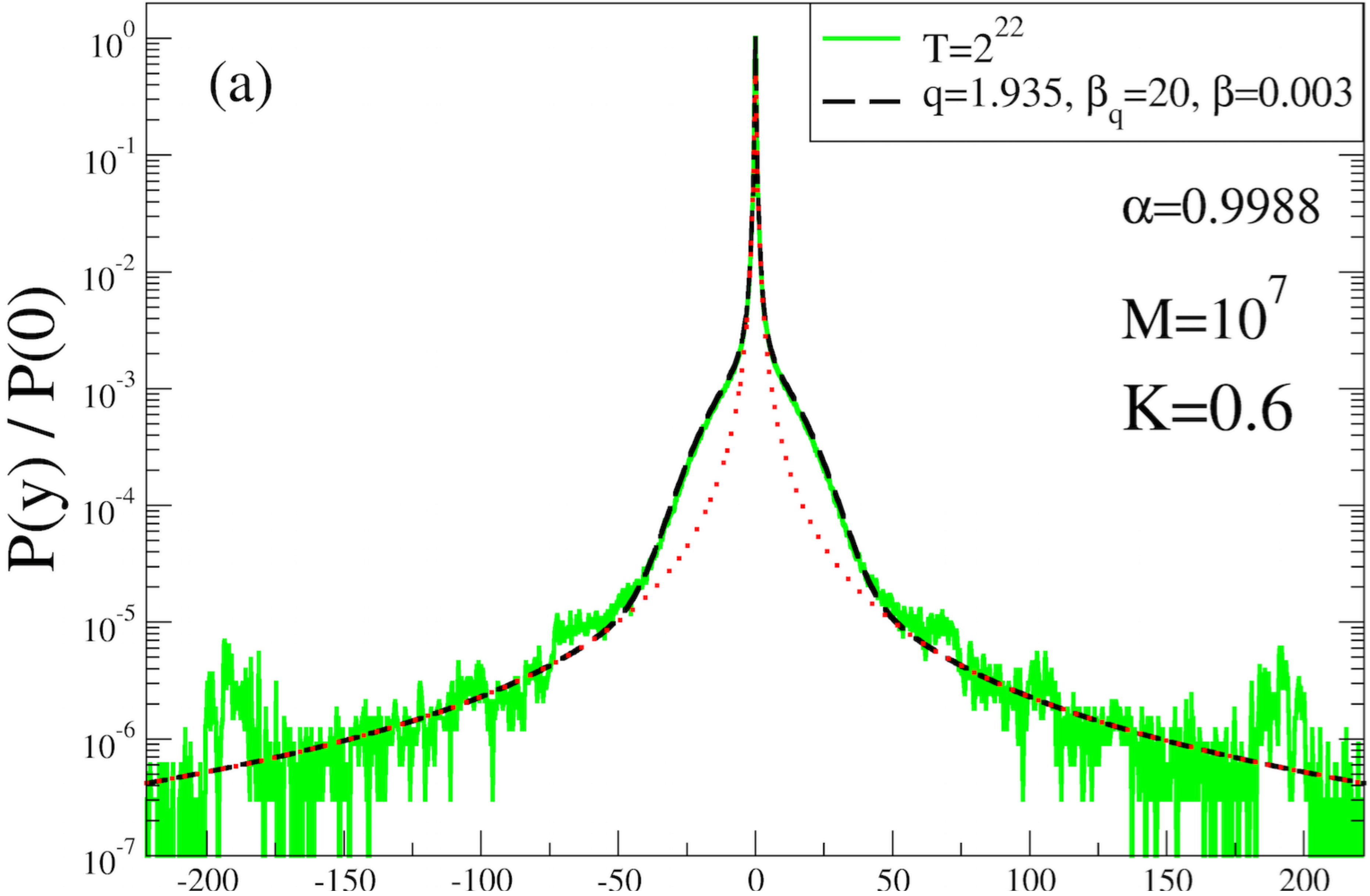


Tirnakli and Borges

Nature / Scientific Reports **6**, 23644 (2016)



Standard Map
30x10^6 random ini. conds.
q=1.935
K=0
P(y)
y
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
-3000
-2000
-1000
0
1000
2000
3000
U. Tirnakli and C. Antonopoulos (2017)



$$\frac{P(y)}{P(0)} = \alpha \; \exp_q(-\beta_q[yP(0)]^2) + (1-\alpha)\exp(-\beta[yP(0)]^2)$$

The standard map: From Boltzmann-Gibbs statistics to Tsallis statistics

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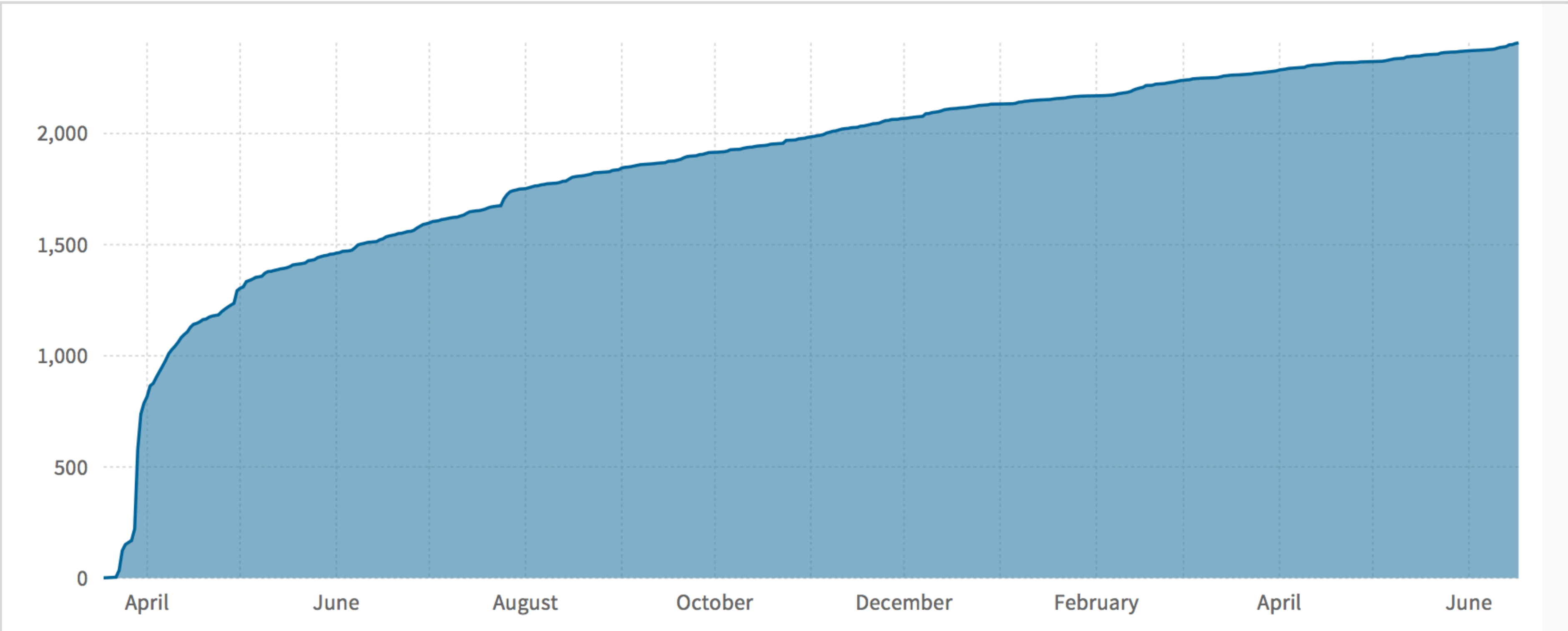


OVERDAMPED MOTION OF REPULSIVELY INTERACTING VORTICES IN TYPE II SUPERCONDUCTORS

PRL **105,** 260601 (2010) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2010

Thermostatistics of Overdamped Motion of Interacting Particles

J. S. Andrade, Jr.,[1,3] G. F. T. da Silva,[1] A. A. Moreira,[1] F. D. Nobre,[2,3] and E. M. F. Curado[2,3]

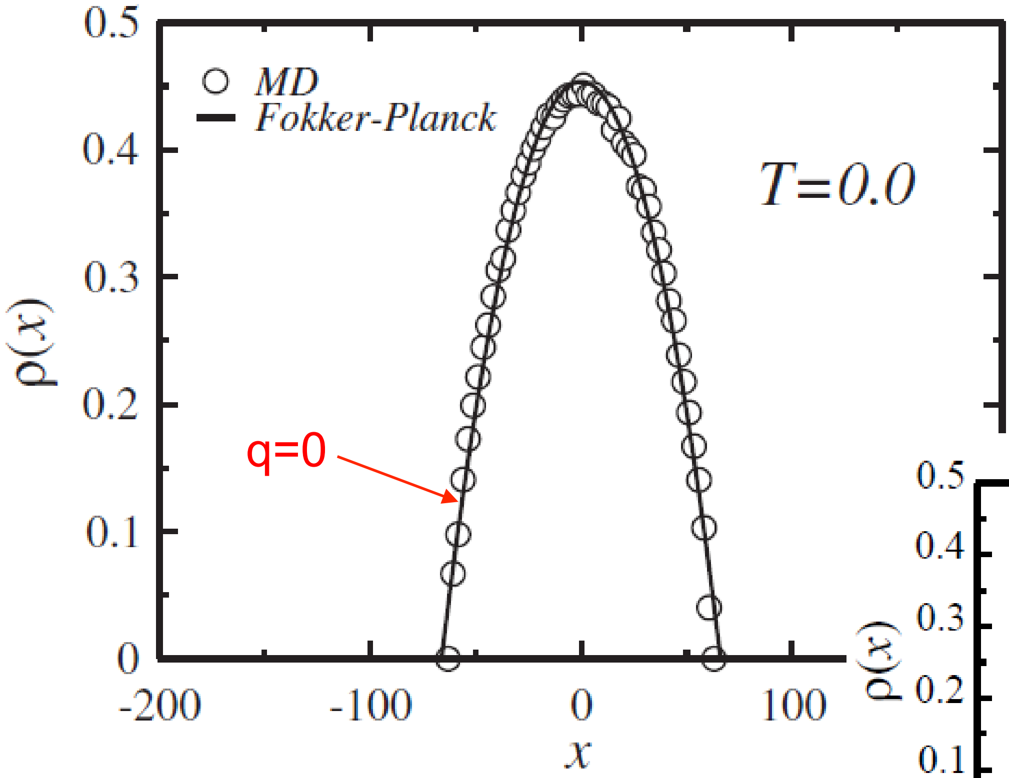
[1]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*

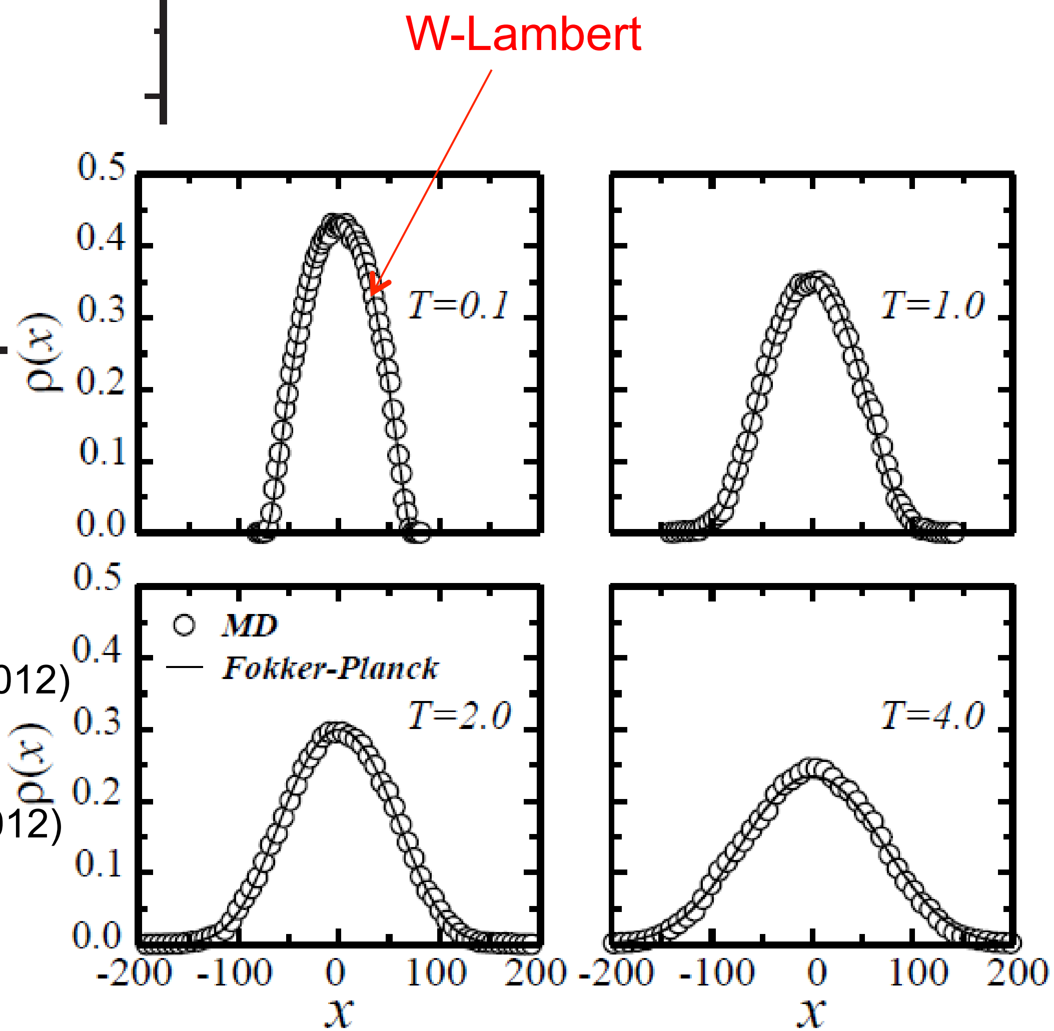
[2]*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180, Rio de Janeiro-RJ, Brazil*

[3]*National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180, Rio de Janeiro-RJ, Brazil*

(Received 8 August 2010; published 22 December 2010)

We show through a nonlinear Fokker-Planck formalism, and confirm by molecular dynamics simulations, that the overdamped motion of interacting particles at $T = 0$, where T is the temperature of a thermal bath connected to the system, can be directly associated with Tsallis thermostatistics. For sufficiently high values of T, the distribution of particles becomes Gaussian, so that the classical Boltzmann-Gibbs behavior is recovered. For intermediate temperatures of the thermal bath, the system displays a mixed behavior that follows a novel type of thermostatistics, where the entropy is given by a linear combination of Tsallis and Boltzmann-Gibbs entropies.





See also:

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Ribeiro, Nobre and Curado, PRE **85**, 121046 (2012)

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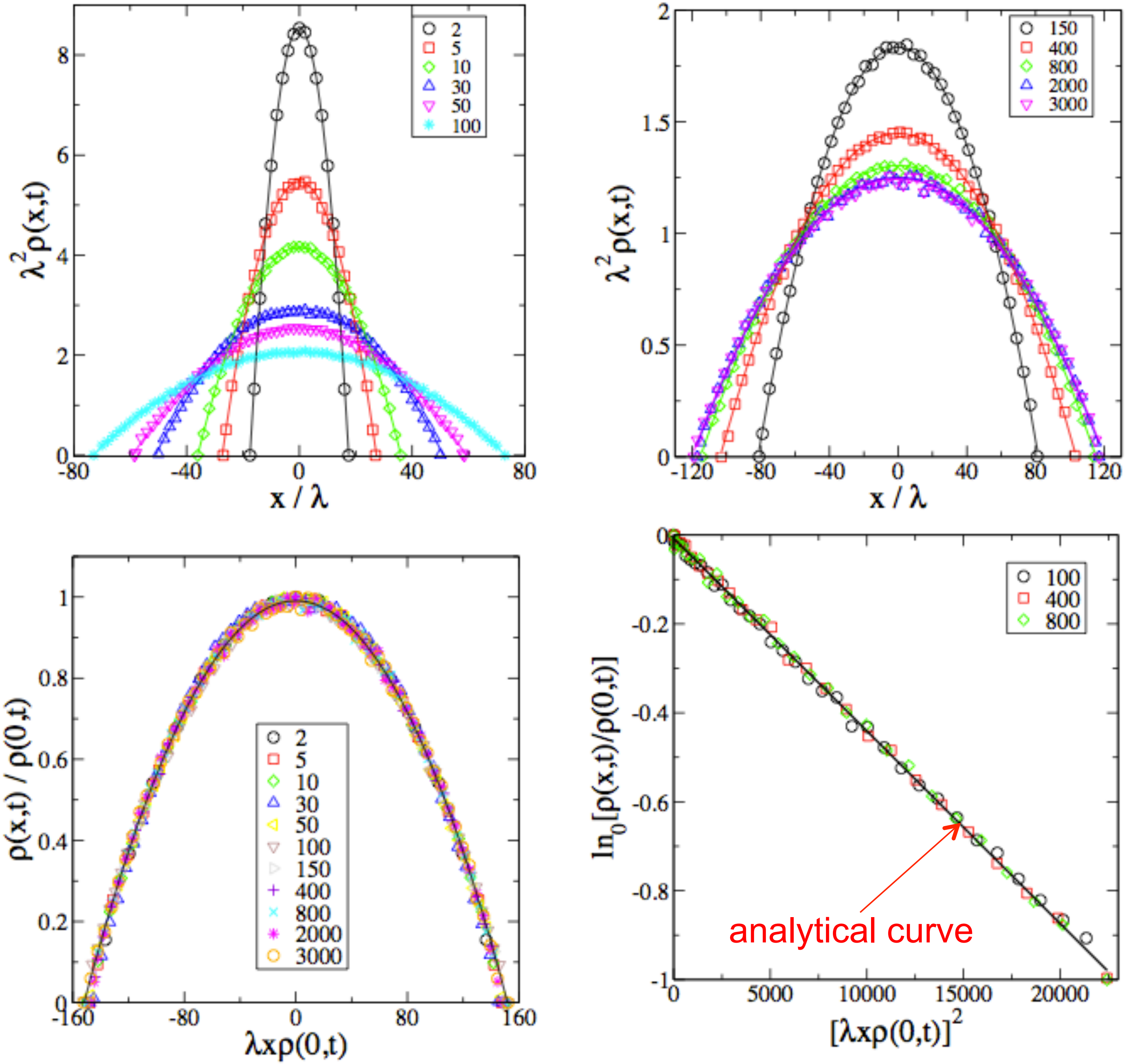
PHYSICAL REVIEW E **85**, 021146 (2012)

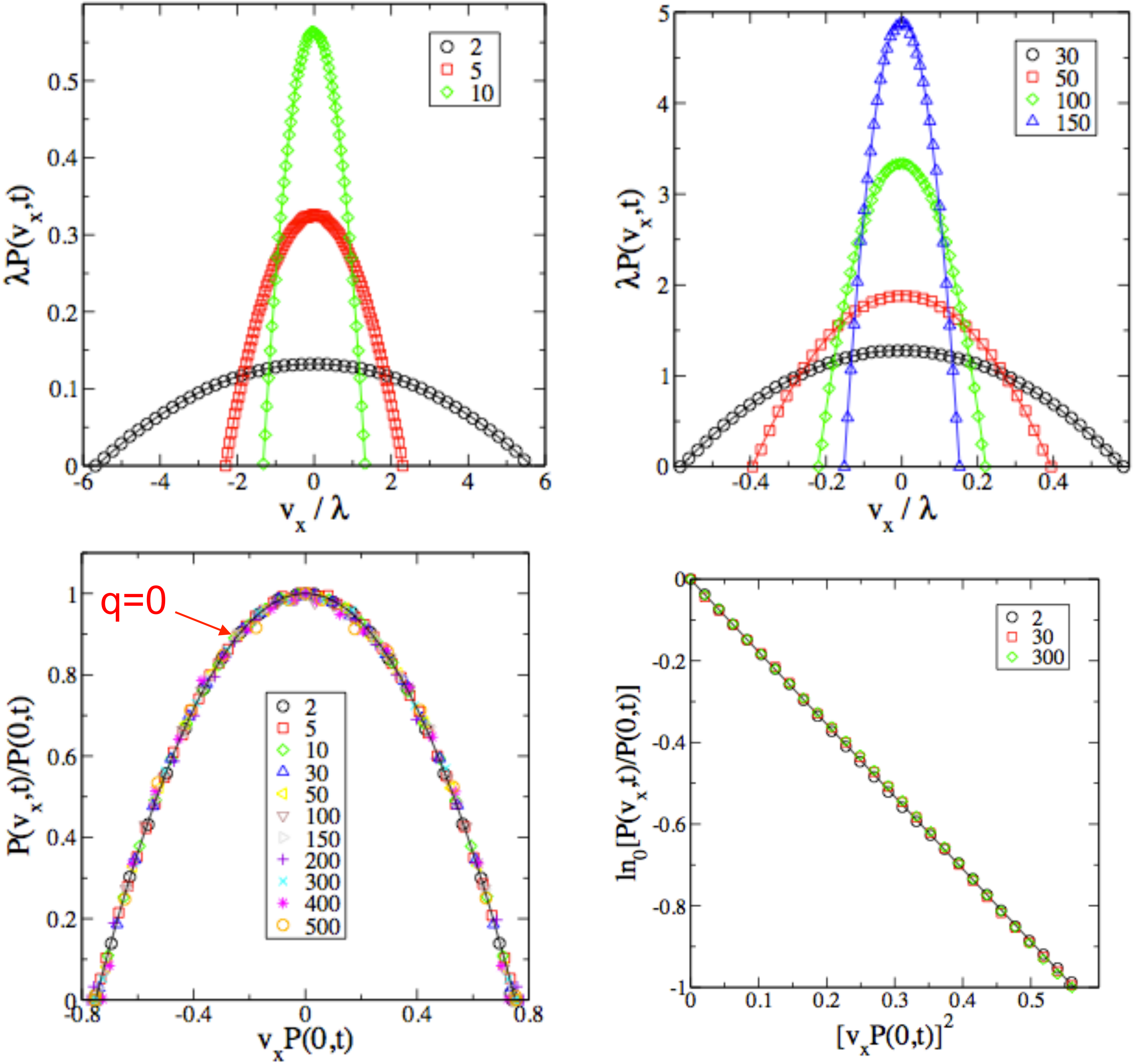
Time evolution of interacting vortices under overdamped motion

Mauricio S. Ribeiro,[1,*] Fernando D. Nobre,[1,†] and Evaldo M. F. Curado[1,2,‡]
[1]*Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, Rio de Janeiro, RJ 22290-180, Brazil*
[2]*Laboratoire APC, Université Paris Diderot, 10 rue A. Domon et L. Duquet, 75205 Paris, France*
(Received 15 December 2011; published 27 February 2012)

A system of interacting vortices under overdamped motion, which has been commonly used in the literature to model flux-front penetration in disordered type-II superconductors, was recently related to a nonlinear Fokker-Planck equation, characteristic of nonextensive statistical mechanics, through an analysis of its stationary state. Herein, this connection is extended by means of a thorough analysis of the time evolution of this system. Numerical data from molecular-dynamics simulations are presented for both position and velocity probability distributions $P(x,t)$ and $P(v_x,t)$, respectively; both distributions are well fitted by similar q-Gaussian distributions, with the same index $q = 0$, for all times considered. Particularly, the evolution of the system occurs in such a way that $P(x,t)$ presents a time behavior for its width, normalization, and second moment, in full agreement with the analytic solution of the nonlinear Fokker-Planck equation. The present results provide further evidence that this system is deeply associated with nonextensive statistical mechanics.



MS Ribeiro, FD Nobre and EMF Curado, PRE **85**, 121046 (2012)



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